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# A dual relation between port-Hamiltonian systems and the Brayton–Moser equations for nonlinear switched RLC circuits<sup>☆</sup>

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## Abstract

In the last decades, several researchers have concentrated on the dynamic modeling of nonlinear electrical circuits from an energy-based perspective. A recent perspective is based on the concept of port-Hamiltonian (PH) systems. In this paper, we discuss the relations between the classical Brayton–Moser (BM) equations—stemming from the early sixties—and PH models for topologically complete nonlinear RLC circuits, with and without controllable switches. It will be shown that PH systems precisely dualize the BM equations, leading to possible advantages at the level of controller design. Consequently, useful and important properties of the one framework can be translated to the other. Control designs for the PH model cannot be directly implemented since they require observation of flux and charges, which are not directly available through standard sensors, while the BM models require only observation of currents and voltages. The introduced duality allows to pull back PH designs to the space of currents and voltages. This offers the possibility to exchange several different techniques, available in the literature, for modeling, analysis and controller design for RLC circuits. Illustrative examples are provided to emphasize the duality between both frameworks.

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**Keywords:** Nonlinear networks; Brayton–Moser equations; Hamiltonian systems; Power converters; Switched-mode circuits

“Black fish blue fish old fish new fish.” (Dr. Seuss, 1960)

## 1. Introduction

From the early 1960s until the early 1980s many researchers have concentrated on the development of systematic tools for the formulation of the dynamic behavior of nonlinear electrical circuits. Most of these works have in common that the methods are based on the use of the energy, power and the topological properties of the system. Pioneering results were reported by e.g. Brayton and Moser (1964) and MacFarlane (1970). Their method is mainly based on the definition of some mixed-potential function having the unit of power. Another approach was considered by

Chua and McPherson (1974). Their method used the classical Lagrangian framework, but the choice of coordinates departed radically from conventional thinking. Almost a decade later, in Kwatny, Massimo, and Bahar (1982) a generalized Lagrangian framework is proposed in which some severe limitations of the previous methods are relaxed. After this period the area became relatively quiet, until recently with the introduction of port-Hamiltonian (PH) systems (van der Schaft, 2000) and Lagrangian modeling of power electronic systems in Ortega, Loria, Nicklasson, and Sira-Ramírez (1998) and Scherpen, Jeltsema, and Klaassens (2003). In the context of switched-mode systems it is shown that the dynamics correspond to systems derivable from a Lagrangian or PH point of view. This has the advantage that control techniques, like passivity-based control, can be successfully applied to such circuits.

In this paper we will concentrate on two specific formulations: systems based on the classical Brayton–Moser (BM) equations and PH systems. In view of its practical applications related to controller design, we want to establish a connection between the two formalisms and discuss their advantages and disadvantages. The most trivial ‘duality’

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between the two frameworks for electrical circuits is that PH systems assume the circuit elements to be flux and charge controlled only, while the BM equation impose the restriction that the elements are controlled by the dual variables, namely current and voltage, respectively. If the frameworks are used to design feedback controllers, the controller will consequently rely on some output or state measurements, i.e., measurements of fluxes and charges or currents and voltages. In a practical situation the off-the-shelf available sensors give as output the measurements in terms of current or voltage quantities only. In the linear case the relation between flux and current or charge and voltage is an easily obtained bijective relation, but if a system contains highly nonlinear elements complicated state transformations have to be included or quality degrading approximations have to be made. Since in general the elements may not have bijective relations, even more serious problems may arise.

One reason to work with PH systems is that the dynamic equations are formulated in physical or ‘natural’ variables. In case of autonomous LC circuits this can be considered a reasonable argument, but, on the other hand, the inclusion of converter elements, like sources and resistors seems not so natural in the PH framework. In principal, the constitutive relations of controlled voltage sources, current sources and dissipative elements are rather considered in terms of currents or voltages, instead of fluxes or charges, see e.g. MacFarlane (1970). It then seems to be more natural to choose for the BM formalism. Therefore, it is of interest to study if there exists some fundamental relation, in a mathematical sense, between both frameworks. Indeed, as will be shown throughout the paper, under some reasonable assumptions such a relation exists. As a consequence, essential and important properties of one framework can be translated to the other. For example, the established relations enable us to translate (already available) controller structures from one framework to the other.

The paper is organized as follows. In Section 2, we briefly recall the definition of the BM equations and PH systems. In Section 3, the connections between both frameworks are first established for autonomous LC circuits. Section 4 deals with the concept of implicit PH systems. This concept is then to be translated to the BM equations. As a result, we provide a novel procedure to obtain minimal state space representations of circuits containing inductor-only cutsets and/or capacitor-only loops. The section ends with an illustrative example. Finally, we derive the dissipation and power-supply part of the mixed-potential function for the BM equations from an PH external port point of view. Both frameworks are compared in the presence of power sources and resistive elements and (controllable) switches.

## 2. Autonomous LC circuits

In this section we briefly recall both the concept of PH systems and the BM equations for nonlinear topologically

complete LC circuits without non-energetic elements such as resistors and sources. A circuit is called ‘topologically complete’ if it can be described by an independent set of inductor currents and capacitor voltages such that Kirchhoff’s laws are satisfied. For a detailed treatment on topologically completeness, the reader is referred to Weiss, Mathis, and Trajkovic (1998). Apart from topologically completeness, we also restrict the discussion to circuits without any elements in excess, i.e., we do not admit inductor-only cutsets and/or capacitor-only loops. This condition will be relaxed in Section 4.

### 2.1. Brayton and Moser’s equations

Consider a (possibly nonlinear) electrical circuit with  $\rho$  capacitors and  $\sigma$  inductors. The order  $n$  of the circuit is defined by  $n = \rho + \sigma$ . Let  $v_C = \text{col}(v_{C_1}, \dots, v_{C_\rho}) \in \mathbb{M}_C$  and  $i_L = \text{col}(i_{L_1}, \dots, i_{L_\sigma}) \in \mathbb{M}_I$  denote the voltage across the capacitors and the currents through inductors, respectively. By  $\mathbb{M}_C$  we denote the space of capacitor voltages and by  $\mathbb{M}_I$  the space of inductor currents, where  $\mathbb{M}_C = \mathbb{R}^\rho$  and  $\mathbb{M}_I = \mathbb{R}^\sigma$ . The full state-space is defined by  $\mathbb{M} = \mathbb{M}_C \times \mathbb{M}_I = \mathbb{R}^n$ . Furthermore, let  $q_C = \text{col}(q_{C_1}, \dots, q_{C_\rho}) \in \mathbb{V}_C$  represent the charges on the capacitor plates and  $\varphi_L = \text{col}(\varphi_{L_1}, \dots, \varphi_{L_\sigma}) \in \mathbb{V}_I$  represent the fluxes in the inductor coils, where  $\mathbb{V}_C = \mathbb{R}^\rho$  and  $\mathbb{V}_I = \mathbb{R}^\sigma$  denote the spaces of charges and fluxes, with  $\mathbb{V} = \mathbb{V}_C \times \mathbb{V}_I = \mathbb{R}^n$ . Under the assumption that the capacitors are voltage-controlled and the inductors are current-controlled, i.e., there exist smooth functions  $\hat{q}_C: \mathbb{M}_C \rightarrow \mathbb{V}_C$ , and  $\hat{\varphi}_L: \mathbb{M}_I \rightarrow \mathbb{V}_I$ , describing the constitutive co-relations of the capacitors and the inductors, respectively, it is shown in Brayton and Moser (1964) that the dynamical behavior of such circuits is governed by the following set of differential equations

$$\begin{aligned} C(v_C) \frac{dv_C}{dt} &= \frac{\partial \mathcal{P}_T}{\partial v_C}(v_C, i_L), \\ -L(i_L) \frac{di_L}{dt} &= \frac{\partial \mathcal{P}_T}{\partial i_L}(v_C, i_L), \end{aligned} \quad (1)$$

where  $C(v_C) = (\partial/\partial v_C)\hat{q}_C(v_C)$  and  $L(i_L) = (\partial/\partial i_L)\hat{\varphi}_L(i_L)$  denote the  $\rho \times \rho$  capacitor and  $\sigma \times \sigma$  inductor matrices, respectively. The scalar function  $\mathcal{P}_T: \mathbb{M} \rightarrow \mathbb{R}$ , called the *mixed-potential*, which, in case there are no resistors and sources, is proportional to the power circulating across the capacitors or the inductors in the circuit. The form of the mixed-potential function directly follows from application of Kirchhoff’s laws. For an autonomous LC circuit Kirchhoff’s laws read

$$\begin{aligned} \frac{dq_C}{dt} - \psi^\top i_L &= 0, \\ \frac{d\varphi_L}{dt} + \psi v_C &= 0, \end{aligned} \quad (2)$$

where  $(d/dt)q_C \in \mathbb{V}_C$  represent the currents through the capacitors and  $(d/dt)\varphi_L \in \mathbb{V}_I$  represent the voltages across the

inductors, respectively. Furthermore,  $\psi \in \mathbb{R}^{\rho \times \sigma}$  is a constant matrix reflecting the interconnection structure of the  $L$  and  $C$  elements. Hence, by using  $(d/dt)q_C = C(v_C)(d/dt)v_C$  and  $(d/dt)\phi_L = L(i_L)(d/dt)i_L$  it is directly observed from (1) and (2) that the mixed-potential should read as

$$\mathcal{P}_T(v_C, i_L) = i_L^\top \psi v_C. \quad (3)$$

Notice that the mixed-potential precisely coincides with the power circulating across the capacitors  $P_C = v_C^\top (d/dt)q_C$  or minus the power in the inductors  $P_L = i_L^\top (d/dt)\phi_L = -i_L^\top \psi v_C$ .

**Remark 1.** Due to the fact that we have assumed that there are no inductor cutsets and/or capacitor loops (excess elements),  $C(v_C)$  and  $L(i_L)$  are diagonal and the nonlinearity of the separate elements will in many situations depend only on one coordinate, i.e.,  $C_k = C_k(v_{C_k})$ ,  $k \in \rho$  and  $L_j = L_j(i_{L_j})$ ,  $j \in \sigma$ . As we will see later on, excess elements and circuits with coupled-inductors can also be included in the analysis.

## 2.2. Port-Hamiltonian systems

As is well-known from the literature, e.g., van der Schaft (2000), a topologically complete RLC circuit can be thought of as an energy-conserving LC circuit with ports corresponding to the various non-energetic elements. The remaining LC circuit can be represented in an intrinsic way as a Hamiltonian system with port variables. For that reason we refer to such description as a port-Hamiltonian (PH) system. Under the assumption that there exists smooth functions  $\hat{v}_C : \mathbb{V}_c \rightarrow \mathbb{M}_c$  denoting the constitutive relations of the capacitors and  $\hat{i}_L : \mathbb{V}_l \rightarrow \mathbb{M}_l$  denoting the constitutive relations of the inductors, the equations of motion of the circuit are determined by

$$\begin{aligned} \frac{dq_C}{dt} &= \psi^\top \frac{\partial \mathcal{H}}{\partial \phi_L}(q_C, \phi_L), \\ \frac{d\phi_L}{dt} &= -\psi \frac{\partial \mathcal{H}}{\partial q_C}(q_C, \phi_L). \end{aligned} \quad (4)$$

Here  $\mathcal{H} : \mathbb{V} \rightarrow \mathbb{R}$  is a smooth function, called the *Hamiltonian*, representing the total energy stored in the circuit, i.e.,  $\mathcal{H}(q_C, \phi_L) = \mathcal{V}(q_C) + \mathcal{T}(\phi_L)$ , where

$$\mathcal{V}(q_C) = \int_0^{q_C} \hat{v}_C(q'_C) dq'_C$$

is the total stored electric energy in the capacitors, while

$$\mathcal{T}(\phi_L) = \int_0^{\phi_L} \hat{i}_L(\phi'_L) d\phi'_L$$

is the total stored magnetic energy in the inductors. The matrix  $\psi$  is precisely the same as defined under (2) in the previous subsection. For ease of notation, equations (4) can be compactly written as

$$\frac{dz}{dt} = J \frac{\partial \mathcal{H}}{\partial z}(z) \quad (5)$$

with  $z = \text{col}(q_{C_1}, \dots, q_{C_\rho}, \phi_{L_1}, \dots, \phi_{L_\sigma}) \in \mathbb{V}$  and

$$J = \begin{bmatrix} 0 & \psi^\top \\ -\psi & 0 \end{bmatrix}. \quad (6)$$

The matrix  $J$  represents a Dirac structure (van der Schaft, 2000) on the state-space  $\mathbb{V}$  and clearly satisfies the important property  $J = -J^\top$  (skew-symmetry).

**Remark 2.** For circuits without switches  $\psi$  is a constant full-rank matrix, while for circuits containing one or more switches  $\psi$  may not be constant and may not remain full-rank. We come back to this later on.

Notice that both frameworks impose inherent limitations, i.e., the PH framework assumes the capacitors to be only charge-controlled and the inductors to be only flux-controlled, while the BM equations are restricted to voltage- and current-controlled elements. In the following section we show that the frameworks bear an interesting similarity in structure. As before, the analysis is first carried out for (possibly nonlinear) autonomous LC circuits. Before that, we need to make the following assumption.

## 2.3. Main assumption

In order to be able to relate the BM equations and the PH framework, we impose the assumption that all capacitors can be both voltage- or charge-controlled, and that all inductors can be both flux- or current-controlled.

**Assumption 3.** Throughout the document it is assumed that the two sets  $\mathbb{M}$  and  $\mathbb{V}$  are bijective and continuously differentiable. This means that the mappings  $\hat{v}_C : \mathbb{V}_c \rightarrow \mathbb{M}_c$  and  $\hat{i}_L : \mathbb{V}_l \rightarrow \mathbb{M}_l$  admit a mapping  $\hat{q}_C : \mathbb{M}_c \rightarrow \mathbb{V}_c$  and  $\hat{\phi}_L : \mathbb{M}_l \rightarrow \mathbb{V}_l$ , respectively, such that  $\hat{v}_C \circ \hat{q}_C = \hat{q}_C \circ \hat{v}_C = I_\rho$  and  $\hat{i}_L \circ \hat{\phi}_L = \hat{\phi}_L \circ \hat{i}_L = I_\sigma$ , where  $I$  denotes the identity matrix.

## 3. Relation between the BM and PH equations

We are now ready to establish the relation between the BM equations and PH systems. In the present study our perspective is to view the capacitor charges,  $q_C$ , and the inductor fluxes,  $\phi_L$ , as the *energy variables*. Consequently, the capacitor voltages,  $v_C$ , and the inductor currents,  $i_L$ , are referred to as the *co-energy variables*. Our first observation is that the relation between the energy and the co-energy variables is given by

$$\frac{dq_C}{dt} = \psi^\top i_L \quad \text{and} \quad \frac{d\phi_L}{dt} = -\psi v_C. \quad (7)$$

Notice that (7) is precisely (2), where  $\mathbb{V}$  is mapped onto  $\mathbb{M}$  through a differentiation on  $\mathbb{V}$ . Furthermore, using the fact that  $C(v_C) = (\partial/\partial v_C)\hat{q}_C(v_C)$  and  $L(i_L) = (\partial/\partial i_L)\hat{\phi}_L(i_L)$  we may relate a function  $\mathcal{H}^* : \mathbb{M} \rightarrow \mathbb{R}$  with the BM

equations (1) as follows. Let

$$\mathcal{V}^*(v_C) = \int_0^{v_C} \hat{q}_C(v'_C) dv'_C$$

be the total stored electric co-energy, and

$$\mathcal{T}^*(i_L) = \int_0^{i_L} \hat{\phi}_L(i'_L) di'_L$$

the total stored magnetic co-energy in the circuit, then  $\mathcal{H}^*(v_C, i_L) = \mathcal{V}^*(v_C) + \mathcal{T}^*(i_L)$  denotes the total stored co-energy in the circuit. Hence, we may replace the capacitor and inductor matrices in (1) by

$$C(v_C) = \frac{\partial^2 \mathcal{H}^*}{\partial v_C^2}(v_C, i_L) \quad \text{and} \quad L(i_L) = \frac{\partial^2 \mathcal{H}^*}{\partial i_L^2}(v_C, i_L),$$

respectively, and, as a result, the BM equations can be rewritten in a more compact form

$$\frac{d}{dt} \frac{\partial \mathcal{H}^*}{\partial x}(x) = \Phi \frac{\partial \mathcal{P}_T}{\partial x}(x) \quad (8)$$

with  $x = \text{col}(v_{C_1}, \dots, v_{C_\rho}, i_{L_1}, \dots, i_{L_\sigma})$  and  $\Phi = \text{diag}(I_\rho, -I_\sigma)$ . At this point it is interesting to remark that in theoretical mechanics  $\mathcal{H}^*(v_C, i_L)$  is often referred to as the *co-Hamiltonian*.

Let us next study the mathematical relation between (5) and (8). The first, and most obvious, relation is that the Hessian of  $\mathcal{P}_T(x)$  coincides (up to a sign) with the Dirac structure  $J$ . Furthermore, multiplication of (8) with the Dirac structure (6) yields after a few simple manipulations

$$\begin{aligned} \psi^\top \frac{di_L}{dt} &= -\psi^\top \left[ \frac{\partial \hat{\phi}_L}{\partial i_L}(i_L) \right]^{-1} \psi v_C \\ \psi \frac{dv_C}{dt} &= \psi \left[ \frac{\partial \hat{q}_C}{\partial v_C}(v_C) \right]^{-1} \psi^\top i_L. \end{aligned} \quad (9)$$

By Assumption 3,  $[(\partial/\partial v_C)\hat{q}_C(v_C)]^{-1} = (\partial/\partial q_C)\hat{v}_C(q_C)$  and  $[(\partial/\partial i_L)\hat{\phi}_L(i_L)]^{-1} = (\partial/\partial \phi_L)\hat{i}_L(\phi_L)$  are well defined. Finally, by using (7), we obtain a system of second order differential equations in terms of the capacitor charges and inductor fluxes given by

$$\begin{aligned} \frac{d^2 q_C}{dt^2} &= \psi^\top \frac{\partial^2 \mathcal{H}}{\partial q_C^2}(q_C, \phi_L) \dot{\phi}_L, \\ \frac{d^2 \phi_L}{dt^2} &= -\psi \frac{\partial^2 \mathcal{H}}{\partial \phi_L^2}(q_C, \phi_L) \dot{q}_C. \end{aligned} \quad (10)$$

Comparing (10) with the PH equations (4), it is easily recognized that (9), in terms of charges and fluxes, is precisely the time-derivatived (or ‘lifted’) version of the PH equations. Thus, the PH equation can be obtained by multiplication of the BM equations with the Dirac structure and taking the integral with respect to time. Of course, the most direct relation is that both formalisms describe the Kirchhoff laws. Furthermore, the Hamiltonian is defined as the sum of the total electric and magnetic energy in terms of the

energy variables  $q_C$  and  $\phi_L$ , while the co-Hamiltonian is defined in terms of the voltages and currents. As direct consequence of Assumption 3, the relation between the energy and co-energy is defined through a full Legendre transformation as follows. Consider the spaces  $\mathbb{M}$  and  $\mathbb{V}$  with local coordinates  $x$  and  $z$ , respectively. Recall that  $\mathcal{H}(z)$  defines a mapping from  $\mathbb{V}$  to  $\mathbb{R}$ , and  $\mathcal{H}^*(x)$  defines a mapping from  $\mathbb{M}$  to  $\mathbb{R}$ . Then, the Legendre transformation (Arnold, 1989) of the co-Hamiltonian  $\mathcal{H}^*(x)$  is the function  $\mathcal{H}(z)$  of the energy variables  $z$  defined by the equality  $\mathcal{H}(z) = \mathcal{E}(z, \hat{x}(z)) = \max_{x \in \mathbb{M}} \mathcal{E}(z, x)$ , where

$$\mathcal{E}(z, x) = x^\top z - \mathcal{H}^*(x) \quad (11)$$

and  $z = (\partial/\partial x)\mathcal{H}^*(x)$ , with  $z: \mathbb{M} \rightarrow \mathbb{V}$ . Notice that  $\mathcal{H}(z)$  defines a map from  $\mathbb{V}$  to  $\mathbb{R}$  since  $\hat{x}(z) = [(\partial/\partial x)\mathcal{H}^*(x)]^{-1}$  maps  $\mathbb{V}$  to  $\mathbb{M}$  and  $\mathcal{E}(z, x)$  is a map on  $\mathbb{M}$ . Summarizing, we have shown:

**Proposition 4.** *Under Assumption 3, the port-Hamiltonian equations (5), with Hamiltonian the total energy stored in the circuit, dualize<sup>1</sup> the Brayton–Moser equations given in (8), with co-Hamiltonian expressing the total co-energy.*

Notice that Eqs. (8) can be rewritten such that it establishes a port- co-Hamiltonian (PH\*) framework with a Dirac structure given by

$$J = \Phi \frac{\partial^2 \mathcal{P}_T}{\partial x^2}(x). \quad (12)$$

**Remark 5.** In this section we have shown that for topologically complete electrical circuits it is possible to go from a co-Hamiltonian to a Hamiltonian formulation under the condition that the Legendre transformation is *injective*. In case the Legendre transformation is *surjective*, then every co-Hamiltonian formulation can be obtained from a Hamiltonian one. However, in order to go from one formulation to the other we need the bijectivity conditions as stated in Assumption 3.

#### 4. Excess elements: implicit systems

In this section we extend the latter formulations to LC networks which contain capacitor-only loops and/or inductor-only cutsets. In Kwatny et al. (1982) it is stated that the excess elements do not contribute extra state variables to the formulation, but they do contribute extra co-energy terms to their Lagrangian. Consequently, the order of the network is not simply  $n = \rho + \sigma$ . Although the method of Kwatny et al. (1982) seems the most direct and simple one when dealing with Lagrangian dynamics, it becomes much more involved for Hamiltonian systems, especially when the constitutive relations are nonlinear. Moreover, as will

<sup>1</sup> Dual in the sense of the coordinates spaces and related energy storage properties.



become clear later on, in many cases the physical interpretation of the coordinates will be lost. Here we propose an alternative method which is based on the introduction of implicit systems using Lagrange multipliers.

In the context of mechanical systems it is well-known (van der Schaft, Dalsmo, & Maschke, 1996) that the kinematic constraints can be expressed as  $A^\top(q)\dot{q} = 0$ , with  $\dot{q} \in \mathbb{R}^m$  the vector of generalized velocities and  $A^\top(q)$  some  $m \times k$  matrix of rank  $k$ . The corresponding constraint forces are of the form  $A^\top(q)\lambda$ , where the Lagrange multipliers  $\lambda \in \mathbb{R}^m$  are determined by the requirement that the constraints  $A^\top(q)\dot{q} = 0$  need to be satisfied at all time. If we transform the latter properties to the electrical domain, the corresponding PH equations become

$$\begin{aligned} \frac{d\tilde{z}}{dt} &= \tilde{J} \frac{\partial \tilde{\mathcal{H}}(\tilde{z})}{\partial \tilde{z}} + A(\tilde{z})\lambda, \\ 0 &= A^\top(\tilde{z}) \frac{\partial \tilde{\mathcal{H}}(\tilde{z})}{\partial \tilde{z}}, \end{aligned} \quad (13)$$

where we denote  $\tilde{z} \in \tilde{\mathcal{V}} = \mathbb{R}^{\tilde{n}}$ ,  $\tilde{J} \in \tilde{\mathcal{V}} \times \tilde{\mathcal{V}}$ , and  $\tilde{\mathcal{H}}: \tilde{\mathcal{V}} \rightarrow \mathbb{R}$  as the augmented energy variables, augmented Dirac structure, and augmented Hamiltonian, respectively. The constraint PH equations are possibly non-minimal in the sense that certain energy variables have to be eliminated first to obtain a minimal representation of order  $n \leq \tilde{n} := \rho + \sigma$ . For electrical circuits without switches  $A$  is constant, i.e.,  $A(\tilde{z}) = A \in \mathbb{R}^{k \times (\rho + \sigma)}$ , with  $k$  the number of independent constraints. Eqs. (13) are often called an *implicit generalized* PH system, for more details see van der Schaft et al. (1996). Before we proceed, we first decompose the constraint matrix  $A$  into

$$A = \begin{bmatrix} A_C & 0 \\ 0 & A_L \end{bmatrix}$$

and the Lagrange multiplier into  $\lambda = \text{col}(\lambda_C, \lambda_L)$ . The matrix  $A_C$  captures Kirchhoff's voltage law applied to the capacitor-only loops, while  $A_L$  captures Kirchhoff's current law applied to the inductor cut-sets in the circuit.

In order to accommodate the use of Lagrange multipliers with the BM equations, (8) must be altered as follows:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \tilde{\mathcal{H}}^*(\tilde{x})}{\partial \tilde{x}} &= \tilde{\Phi} \frac{\partial \tilde{\mathcal{P}}_T}{\partial \tilde{x}}(\tilde{x}, \lambda), \\ 0 &= A^\top \tilde{x}, \end{aligned} \quad (14)$$

where  $\tilde{x} \in \tilde{\mathcal{M}} = \mathbb{R}^{\tilde{n}}$  are the augmented co-energy variables and  $\tilde{\mathcal{P}}_T: \tilde{\mathcal{M}} \rightarrow \mathbb{R}$  is the constraint mixed-potential function given by

$$\tilde{\mathcal{P}}_T(\tilde{v}_C, \tilde{i}_L, \lambda) = \tilde{i}_L^\top \tilde{\psi} \tilde{v}_C + \tilde{v}_C^\top A_C \lambda_C - \tilde{i}_L^\top A_L \lambda_L. \quad (15)$$

Following van der Schaft et al. (1996) and according to Proposition 4, we call (14) an *implicit generalized* PH\* system. The procedure to obtain the equations of motion for (possibly nonlinear) LC networks with or without excess elements can be summarized by the following steps (the brackets  $\{\cdot\}$  refer to the PH case).

*Procedure:*

- (1) Give the  $\rho$  capacitive elements a voltage {charge} coordinate, and the  $\sigma$  inductive elements a current {flux} coordinate,  $v_{C_k} \{q_{C_k}\}$ ,  $k=1, \dots, \rho$ , and  $i_{L_j} \{\varphi_{L_j}\}$ ,  $j=1, \dots, \sigma$ , respectively.
- (2) Determine the corresponding co-energy {energy} for all elements, i.e., the electric co-energy {energy},  $\mathcal{V}^*(v_C) \{\mathcal{V}(q_C)\}$ , for the capacitive elements, and the magnetic co-energy {energy},  $\mathcal{T}^*(i_L) \{\mathcal{T}(\varphi_L)\}$ , for the inductive elements.
- (3) If there exists a capacitor-only loop, apply Kirchhoff's voltage law to the loop to obtain  $A_C$ . If there exists an inductor-only cutset, apply Kirchhoff's current law to this cutset to obtain  $A_L$ . If there are no excess elements, put  $A = 0$ , and go to Step 5. Otherwise, proceed with Step 4.
- (4) Choose which elements are to be referred to as excess elements. Think of them as if they were removed from the circuit, i.e., the excess-capacitors become open circuits, whereas the excess inductors have to be substituted by short circuits. Find the augmented interconnection matrix  $\tilde{\psi}$  by determining the connections between the remaining inductive and capacitive elements by using either Kirchhoff's current law or Kirchhoff's voltage law.
- (5) Plug the information of the previous steps into (14) to obtain the equation of motion. In case of excess elements, the equations of motion are in implicit form and have to be solved with respect to the Lagrange multiplier.

In case the network contains capacitor-only loops or inductor-only cutsets, the coordinates of the resulting elements in excess can be viewed as *intermediate* help-variables. These help-variables are finally removed using the constraint equation. Let us next demonstrate the procedure to obtain a minimal set of equations by studying a simple example.

**Example 6.** Consider the simple LC circuit depicted in Fig. 1. It contains one element in excess arising from the inductor cutset formed by  $\{L_1, L_2, L_3\}$ . For simplicity we assume that all the elements are linear and time-invariant, except for  $L_1$  which is described by the constitutive relation  $\hat{i}_{L_1}(\varphi_{L_1}) = \ell \tanh(\varphi_{L_1})$ . The Hamiltonian for the circuit is

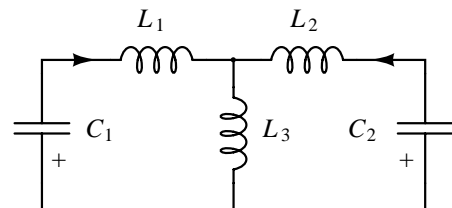


Fig. 1. Simple LC circuit with inductor-only cutset.

defined as

$$\begin{aligned}\tilde{\mathcal{H}}(q_C, \tilde{\varphi}_L) = & \frac{1}{2C_1} q_{C_1}^2 + \frac{1}{2C_2} q_{C_2}^2 + \frac{1}{2L_2} \varphi_{L_2}^2 \\ & + \frac{1}{2L_3} \varphi_{L_3}^2 + \int_0^{\varphi_{L_1}} \hat{i}_{L_1}(\varphi'_{L_1}) d\varphi'_{L_1},\end{aligned}$$

while according to (11) the corresponding co-Hamiltonian should read

$$\begin{aligned}\tilde{\mathcal{H}}^*(v_C, \tilde{i}_L) = & \frac{1}{2} C_1 v_{C_1}^2 + \frac{1}{2} C_2 v_{C_2}^2 + \frac{1}{2} L_2 i_{L_2}^2 \\ & + \frac{1}{2} L_3 i_{L_3}^2 + \int_0^{i_{L_1}} \hat{\varphi}_{L_1}(i'_{L_1}) di'_{L_1},\end{aligned}$$

where  $\hat{\varphi}_{L_1}(i_{L_1}) = \frac{1}{2} [\ln(-i_{L_1} - \ell) - \ln(i_{L_1} - \ell)]$  denotes the constitutive co-relation for  $L_1$ . Let us start by deriving the PH equations for the circuit.

The constraint equation in terms of the fluxes can be expressed as  $\ell \tanh(\varphi_{L_1}) + (\varphi_{L_2}/L_2) - (\varphi_{L_3}/L_3) = 0$ . Hence, the constraint matrix equals  $A = \text{col}(0, 0, 1, 1, -1)$  and  $\lambda = \lambda_L \in \mathbb{R}$ . An appropriate choice is to view  $L_3$  as the excess element and try to eliminate  $\varphi_{L_3}$ . Using either Kirchhoff's current or voltage law (7), the augmented interconnection matrix connecting the remaining coordinates is found as

$$\tilde{\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^\top. \quad (16)$$

Hence, after substituting the latter into (13) yields the equations of motion in implicit PH form (of non-minimal order), i.e.,

$$\dot{q}_{C_1} = \ell \tanh(\varphi_{L_1}), \quad \dot{\varphi}_{L_1} = -\frac{q_{C_1}}{C_1} + \lambda_L,$$

$$\dot{q}_{C_2} = \frac{\varphi_{L_2}}{L_2}, \quad \dot{\varphi}_{L_2} = -\frac{q_{C_2}}{C_2} + \lambda_L,$$

$$\dot{\varphi}_{L_3} = -\lambda_L, \quad 0 = \ell \tanh(\varphi_{L_1}) + \frac{\varphi_{L_2}}{L_2} - \frac{\varphi_{L_3}}{L_3}.$$

Next, in order to eliminate  $\lambda_L$ , we have to use the time-derivative of the constraint equation and use it to replace  $\dot{\varphi}_{L_3}$  as follows:

$$\dot{\varphi}_{L_3} = \ell L_3 \frac{\partial}{\partial \varphi_{L_1}} [\tanh(\varphi_{L_1})] \dot{\varphi}_{L_1} + \frac{L_3}{L_2} \dot{\varphi}_{L_2}$$

yielding the minimal-order circuit equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + \frac{\ell L_3}{\cosh^2(\varphi_{L_1})} & \frac{L_3}{L_2} \\ 0 & 0 & \frac{\ell L_3}{\cosh^2(\varphi_{L_1})} & 1 + \frac{L_3}{L_2} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} q_{C_1} \\ q_{C_2} \\ \varphi_{L_1} \\ \varphi_{L_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{q_{C_1}}{C_1} \\ \frac{q_{C_2}}{C_2} \\ \ell \tanh(\varphi_{L_1}) \\ \frac{\varphi_{L_2}}{L_2} \end{bmatrix}.$$

Notice that  $\psi$  is obtained by extracting the zero rows and columns of (16). After some algebraic manipulations an explicit description is obtained as

$$\dot{q}_{C_1} = \ell \tanh(\varphi_{L_1}),$$

$$\dot{q}_{C_2} = \frac{\varphi_{L_2}}{L_2},$$

$$\dot{\varphi}_{L_1} = \frac{\cosh^2(\varphi_{L_1})}{\ell \Gamma} \left( L_3 \frac{q_{C_2}}{C_2} - (L_2 + L_3) \frac{q_{C_1}}{C_1} \right), \quad (17)$$

$$\dot{\varphi}_{L_2} = \frac{L_2 L_3}{\Gamma} \frac{q_{C_1}}{C_1} - \frac{L_2}{\Gamma} \left( \frac{\cosh^2(\varphi_{L_1})}{\ell} + L_3 \right) \frac{q_{C_2}}{C_2}$$

with  $\Gamma = \ell^{-1}(L_2 + L_3) \cosh^2(\varphi_{L_1}) + L_2 L_3$ .

Next, we derive the dynamics using the implicit BM description (14). The corresponding augmented co-Hamiltonian  $\tilde{\mathcal{H}}^*(v_C, i_L)$ , constraint matrix  $A$  and the augmented interconnection structure are as above. Hence, by substituting the latter into (14), and after eliminating  $\lambda_L$ , we obtain the circuit equations in BM form (8) as

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 + \frac{\ell}{\ell^2 - i_{L_1}^2} & L_3 \\ 0 & 0 & L_3 & L_2 + L_3 \end{bmatrix}}_{=: Q^*(i_{L_1})} \frac{d}{dt} \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_{L_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ -v_{C_1} \\ -v_{C_2} \end{bmatrix}, \quad (18)$$

or equivalently  $Q^*(x)\dot{x} = Jx$ , with  $Q^*(x) = Q^*(i_{L_1})$ . Notice that the latter equations coincide with the PH description (17) by observing that  $i_{L_1} = \ell \tanh(\varphi_{L_1})$  and  $i_{L_2} = L_2^{-1} \varphi_{L_2}$ . On the other hand, the form of the PH equations (17) are not the same as the ones which would be obtained by substituting the constraint equation into the augmented Hamiltonian. We come back to this in a moment.

It is interesting to point out that with this method we have indirectly shown how to define co-energy with the excess elements (called the extra co-energy terms) as proposed in Kwatny et al. (1982). The co-energy matrix  $Q^*(i_{L_1})$  as obtained in (18) is precisely the one as should be defined when

using the method of Kwatny et al. (1982). However, in case of the Hamiltonian dynamics, it can be checked that we can also obtain a solution if we invert the co-energy matrix. From a circuit-theoretic point of view, inversion of the co-energy matrix corresponds to what is known in classical network analysis as  $\Delta$ -Y or Y- $\Delta$  transformation, i.e., for the given example the inductor-only cutset (Y or impedance representation) is transformed into an inductor loop ( $\Delta$  or admittance representation). Although this transformation leads to a set of dynamic equations, the physical information concerning the origin of the variables is lost. To see this, let us consider another example.

**Example 7** (Example 6 cont'd). Consider again the circuit of Fig. 1. The dynamic equations in the form (5) are obtained as follows. Let

$$Q(\varphi_{L_1}) = [Q^*(i_{L_1})]^{-1}|_{i_{L_1} = \ell \tanh(\varphi_{L_1})},$$

where  $Q^*(i_{L_1})$  represents the co-energy matrix as found in the previous example, and define  $Q(\varphi_{L_1}) = Q(z)$  as the energy matrix. Hence, the corresponding PH equations are directly obtained from the BM equations (18), and take the form  $\dot{z}' = JQ(z')z'$ , or explicitly

$$\begin{aligned} \dot{q}'_{C_1} &= \frac{L_2 + L_3}{\Gamma} \varphi'_{L_1} - \frac{L_3}{\Gamma} \varphi'_{L_2}, \\ \dot{q}'_{C_2} &= \frac{\cosh^2(\varphi'_{L_1}) + \ell L_3}{\ell \Gamma} \varphi'_{L_2} - \frac{L_3}{\Gamma} \varphi'_{L_1}, \\ \dot{\varphi}'_{L_1} &= -\frac{q'_{C_1}}{C_1}, \\ \dot{\varphi}'_{L_2} &= -\frac{q'_{C_2}}{C_2}, \end{aligned} \quad (19)$$

where we have introduced  $(\cdot)'$  to be able to distinguish it with the coordinates of Example 6. It is directly observed that the latter equations have a circuit representation as depicted in Fig. 2. Here  $L_a = (\ell \Gamma)^{-1} \cosh^2(\varphi_{L_1})$ ,  $L_b = \Gamma^{-1} L_2$  and  $L_c = \Gamma^{-1} L_3$ . Comparison of (19) with (17) shows that the fluxes of Fig. 2 do not correspond to the separate inductor elements, i.e.,  $\varphi_{L_i} \neq \varphi'_{L_i}$ ,  $i = 1, 2$ , while  $q_{C_i} = q'_{C_i}$ . For example,  $\varphi'_{L_2}$  does not correspond to the inductor  $L_2$ , but to an inductor composed by  $L_1$ ,  $L_2$  and  $L_3$ . The relation between  $\varphi'_{L_2}$  and

$\varphi_{L_2}$  is given by

$$\varphi_{L_2} = \frac{L_2(\cosh^2(\varphi'_{L_1}) + \ell L_3)}{\ell \Gamma} \varphi'_{L_2} - \frac{L_2 L_3}{\Gamma} \varphi'_{L_1}.$$

Similar arguments hold for  $\varphi_{L_1}$ . This shows that the methods of Example 6 are the most physically appealing, in the sense of the coordinate interpretation, when modeling circuits containing excess elements.

**Remark 8.** A pair of magnetically coupled-inductors can be considered as special case of an inductor-only cutset. In Scherpen et al. (2003) a commonly used equivalent representation is used for modeling and analysis purposes. This equivalent representation exists of three inductors forming a Y-circuit followed by an ideal one-to-one transformer to incorporate the galvanic junction. Although there are three inductors, the equations of motion are defined on  $\mathbb{R}^2$ . If we use to method as proposed herein, we need to determine an extra coordinate for the mutual inductance. This intermediate help variable is then eliminated using the constraint equation. With the method used in Scherpen et al. (2003) the mutual inductance does not contribute a coordinate but contributes a crossterm to corresponding magnetic co-energy instead.

## 5. Dissipative elements, sources and switches

In this section we extend the BM equations with dissipative elements and sources in an alternative way in comparison to e.g. Brayton and Moser (1964) and Massimo, Kwatny, and Bahar (1980). It will be shown that these non-energetic elements can be included using an external port point of view as is done in PH systems (van der Schaft, 2000). For simplicity, we will again restrict the developments to circuits that do not contain excess elements. This restriction can be easily relaxed using the developments of the previous section. Finally, we will accommodate the BM equations for the inclusion of controllable switches.

### 5.1. Adding dissipation in PH systems

For sake of completeness, we briefly recall how non-energetic elements are included in the PH framework. Following van der Schaft (2000), we may deduce the external ports from our LC circuit as follows. Suppose there are  $m$  independent resistors in the circuit. Let  $\xi \in \mathbb{R}^m \subseteq \mathbb{M}$  define a set of suitably chosen outputs for the LC circuit and let  $\gamma \in \mathbb{R}^m$  define a set of (control) inputs to be defined, then the PH equations (5) can be extended as

$$\begin{aligned} \dot{z} &= J \frac{\partial \mathcal{H}}{\partial z}(z) + K\gamma, \\ \xi &= K^\top \frac{\partial \mathcal{H}}{\partial z}(z), \end{aligned} \quad (20)$$

where  $K$  is a constant  $n \times m$  matrix selecting the external port variables. As in van der Schaft (2000), the vectors  $\xi$  and

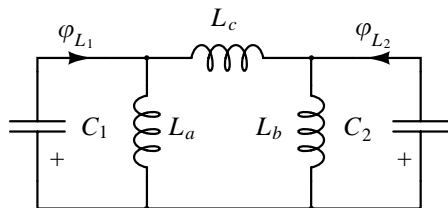


Fig. 2. Admittance ( $\Delta$ ) representation for the circuit of Fig. 1.



$\gamma$  are considered as the power variables at the external ports (voltages and currents) of the circuit that will be terminated by the resistive elements. Termination of these ports may be considered as (static) feedback laws describing the relations between the resistive and dynamic elements. Indeed, let  $S(\xi)$  be a  $m \times m$  smooth matrix function  $S: \mathbb{R}^m \rightarrow \mathbb{R}^m$ , then

$$\gamma = -S(\xi)\xi \quad (21)$$

and hence, after defining

$$R(z) := KS(\xi)|_{\xi=K^\top(\partial/\partial z)\mathcal{H}(z)}K^\top,$$

with  $R(z) = R^\top(z)$  and  $R(0) = 0$ , the PH equations (20) with dissipation become

$$\dot{z} = [J - R(z)] \frac{\partial \mathcal{H}}{\partial z}(z). \quad (22)$$

Voltage and current sources can be included in a similar way by adding terms of the form  $g(z)u(z)$  to the right-hand side of (22), where  $g(z)$  denotes some external interconnection matrix of appropriate dimensions and where the dependence of  $u$  on  $z$  reflects the possibility that the sources are voltage and/or current controlled. Equations (22) are called a port-controlled Hamiltonian system with dissipation (PHD). For a detailed discussion, see van der Schaft (2000).

Notice that (21) constitutes a particular case of Ohm's law yielding that the admissible dissipation in PH systems is restricted to the form

$$R(z) \frac{\partial \mathcal{H}}{\partial z}(z), \quad (23)$$

whereas in general Ohm's law is described by  $\gamma = -\hat{\gamma}(\xi)$ . Also, after applying the feedback, relation (21) is in principle deduced from  $\mathbb{V}$ , i.e., in terms of the charges and the fluxes. Obviously, and especially in the nonlinear case, dissipation in the PH framework seems not so natural because in a practical situation resistors are either voltage or current controlled (see Remark 10).

In the following subsection we will also derive the existence of the mixed-potential function from an external port point of view and show that the admissible dissipation belongs to a slightly broader class than using the PH equations.

### 5.2. Dissipation and BM systems

In a similar fashion as for PH systems in the previous subsection, we may extend (8) in the following alternative way

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{H}^*}{\partial x}(x) &= \Phi \frac{\partial \mathcal{P}_T}{\partial x}(x) + K\gamma, \\ \xi &= K^\top x, \end{aligned} \quad (24)$$

where  $\xi$ ,  $\gamma$  and  $K$  are the same as before. It is seen that the outputs of the LC circuit are directly expressed in terms of the voltages and currents (or at least linear combinations of them) directly. In general resistors are voltage and/or

current controlled. In order to be able to distinguish between these characteristics we proceed by subdividing the external inputs and outputs  $\gamma$  and  $\xi$  as follows. Suppose the circuit contains  $\alpha$  independent voltage-controlled resistors ( $G$ -type) and  $\beta$  independent current-controlled resistors ( $R$ -type). Then  $m = \alpha + \beta$  and the corresponding currents and voltages for the  $G$ -type resistors are given by  $\gamma_G \in \mathbb{R}^\alpha$  and  $\xi_G \in \mathbb{R}^\alpha \subseteq \mathbb{M}_c$ , respectively, and the corresponding voltages and currents for the  $R$ -type resistors are  $\gamma_R \in \mathbb{R}^\beta$  and  $\xi_R \in \mathbb{R}^\beta \subseteq \mathbb{M}_I$ , respectively. Furthermore, let  $\hat{\gamma}_G: \mathbb{M}_c \rightarrow \mathbb{R}^\alpha$  and  $\hat{\gamma}_R: \mathbb{M}_I \rightarrow \mathbb{R}^\beta$  denote the constitutive relations (Ohm's law) of the voltage- and current-controlled resistors, respectively, and define scalar functions  $\bar{\mathcal{P}}_G: \mathbb{R}^\alpha \rightarrow \mathbb{R}$  and  $\bar{\mathcal{P}}_R: \mathbb{R}^\beta \rightarrow \mathbb{R}$  as follows:

$$\bar{\mathcal{P}}_G(\xi_G) = \int_0^{\xi_G} \hat{\gamma}_G(\xi'_G) d\xi'_G, \quad (25)$$

$$\bar{\mathcal{P}}_R(\xi_R) = \int_0^{\xi_R} \hat{\gamma}_R(\xi'_R) d\xi'_R$$

with  $\hat{\gamma}_G(0) = \hat{\gamma}_R(0) = 0$ . Then, termination of the power variables at the external ports yields the relations

$$\gamma_G = -\frac{\partial \bar{\mathcal{P}}_G}{\partial \xi_G}(\xi_G),$$

$$\gamma_R = -\frac{\partial \bar{\mathcal{P}}_R}{\partial \xi_R}(\xi_R).$$

Finally, let  $\bar{\mathcal{P}}_D(\xi)$  be defined as the difference between  $\bar{\mathcal{P}}_G$  and  $\bar{\mathcal{P}}_R$  i.e.,  $\bar{\mathcal{P}}_D(\xi) := \bar{\mathcal{P}}_G(\xi_G) - \bar{\mathcal{P}}_R(\xi_R)$ , then

$$\frac{\partial \bar{\mathcal{P}}_D}{\partial x}(K^\top x) = \frac{\partial^\top \xi}{\partial x} \frac{\partial \bar{\mathcal{P}}_D}{\partial \xi}(\xi) = K \frac{\partial \bar{\mathcal{P}}_D}{\partial \xi}(\xi)$$

and by noting that  $K\gamma = -\Phi^{-1}K(\partial/\partial \xi)\bar{\mathcal{P}}_D(\xi)$  or equivalently  $K\gamma = -\Phi(\partial/\partial x)\bar{\mathcal{P}}_D(K^\top x)$ , with  $\Phi = \Phi^{-1}$ , Eq. (24) becomes

$$\frac{d}{dt} \frac{\partial \mathcal{H}^*}{\partial x}(x) = \Phi \frac{\partial}{\partial x}[\mathcal{P}_T(x) - \mathcal{P}_D(x)], \quad (26)$$

where we have defined  $\mathcal{P}_D(x) := \bar{\mathcal{P}}_D(K^\top x)$ . For circuits without sources and switches we have thus re-derived the form and existence of the dissipative parts of the mixed-potential from an external port point of view and have a procedure to obtain such functions. A similar procedure can be followed in order to include (voltage-controlled) current sources (B-type) and/or (current-controlled) voltage sources (E-type). The only change is that, instead of a negative feedback, now a positive feedback should be constructed due to the natural convention that, unlike (passive) resistors, the sources supply power to the circuit. In that case, the supplied potentials are

$$\begin{aligned} \mathcal{P}_B(\xi_B) &= \int_0^{\xi_B} \hat{\gamma}_B(\xi'_B) d\xi'_B \\ \mathcal{P}_E(\xi_E) &= \int_0^{\xi_E} \hat{\gamma}_E(\xi'_E) d\xi'_E, \end{aligned} \quad (27)$$

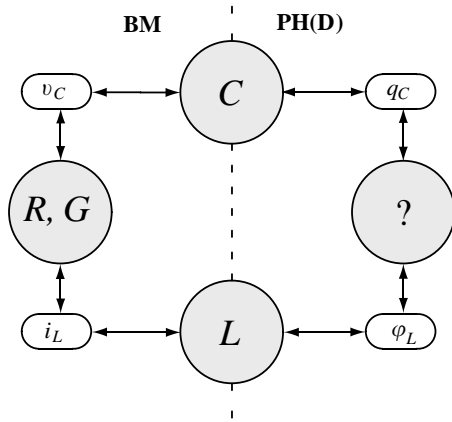


Fig. 3. The two different frameworks: the ‘?’ could be replaced by memristor type of dissipative elements.

respectively, where  $\xi_B \in \mathbb{M}_c$  and  $\xi_E \in \mathbb{M}_l$ . The last step is to define a function  $\mathcal{P}_F(x) := \mathcal{P}_B(x) - \mathcal{P}_E(x)$  and replace in (8),  $\mathcal{P}_T(x)$  by a function  $\mathcal{P}(x)$  defined as  $\mathcal{P}(x) := \mathcal{P}_T(x) - \mathcal{P}_D(x) + \mathcal{P}_F(x)$ , resulting in

$$\frac{d}{dt} \frac{\partial \mathcal{H}^*}{\partial x}(x) = \Phi \frac{\partial \mathcal{P}}{\partial x}(x). \quad (28)$$

Summarizing, we may refer to the BM equations (28) as a co-Hamiltonian framework with dissipation (PH<sup>\*</sup>D).

**Remark 9.** The voltage potential  $\bar{\mathcal{P}}_G(\xi_G)$  is usually referred to as the resistors *co-content*, and the current potential  $\bar{\mathcal{P}}_R(\xi_R)$  is referred to as the resistors *content*, see e.g. MacFarlane (1970) and the references therein.

**Remark 10.** An overall picture of the connection between the BM and PH framework is shown in Fig. 3. It is clear that, in contrast with the PH framework, the BM equations allows to include dissipative elements in a natural way using (the integral version of) Ohm’s law directly. It is of interest to point out that in the nonlinear case the admissible types of dissipation in the PH framework, represented by (23), bear a marked similarity with the definition of the *memristor* as proposed in by Chua in the early seventies (Chua, 1971).<sup>2</sup> However, a detailed discussion is out of place here.

### 5.3. Inclusion of switches

Now that we have explored the connection between both frameworks for nonlinear RLC circuits, we are also interested in accommodating the Brayton–Moser equations for the inclusion of switching functions. The inclusion of switches in the PHD framework is already proposed in Escobar, van der Schaft, and Ortega (1999). For circuits that contain one or more switches, we denote the switch

<sup>2</sup> A memristor is an, yet physically undiscovered, element characterized by a charge-flux curve  $M(q) = (\partial/\partial q)\phi(q)$ , while the corresponding relation between the voltage and current of a memristor is given by  $v_M = M(q)i_M$ , or vice versa. Compare the latter to (22).

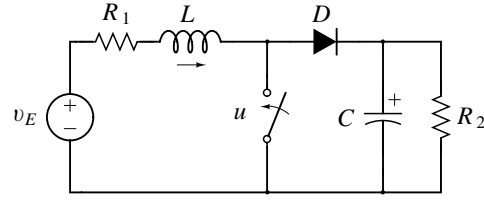


Fig. 4. Single-switch DC-to-DC boost converter.

position(s) by  $u(t) = (u_1(t), \dots, u_m(t))$ , where  $u_j \in U := \{0, 1\}$ ,  $j = 1, \dots, m$ , i.e., ON or OFF, or in other words  $u$  is in the discrete set  $U^m$ . Depending on the application, re-definition of the switching function may also result in for example  $u_j \in U := \{-1, 0, 1\}$ . In case the circuit contains a single controllable switch, the mixed-potential function  $\mathcal{P}(x)$  becomes a switched-mixed-potential function defined as  $\mathcal{P}(x) = \mathcal{P}^u(x)$ , with

$$\mathcal{P}^u(x) = u\mathcal{P}^1(x) + (1 - u)\mathcal{P}^0(x), \quad (29)$$

where  $\mathcal{P}^1(x)$  is the mixed-potential function for the switch position value  $u = 1$ , and  $\mathcal{P}^0(x)$  is the mixed-potential function for the switch in position  $u = 0$ . To see how the various potentials evaluate at different switch positions we consider the following example.

**Example 11.** Consider the single-switch DC-to-DC boost converter depicted in Fig. 4. Assume that the resistors, capacitor and inductor have linear constitutive relations. The active switch  $u$  is the external control input for the circuit, while the diode  $D$  (passive switch) has a smooth nonlinear ( $R$ -type) characteristic  $v_D = \hat{v}_D(i_D)$ . The converter has two stages:  $u = 1$  (switch ON) and  $u = 0$  (switch OFF). The total stored co-energy equals  $\mathcal{H}^*(v_C, i_L) = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2$ , while the corresponding internal potentials for the switch in position  $u = 1$  are readily found as

$$\mathcal{P}_T^1 = 0, \quad \mathcal{P}_G^1 = \frac{1}{2R_2} v_C^2 \quad \text{and} \quad \mathcal{P}_R^1 = \frac{1}{2} R_1 i_L^2.$$

The corresponding internal potentials for  $u = 0$  are

$$\mathcal{P}_T^0 = i_L v_C, \quad \mathcal{P}_G^0 = \frac{1}{2R_2} v_C^2 \quad \text{and}$$

$$\mathcal{P}_R^0 = \frac{1}{2} R_1 i_L^2 + \int_0^{i_L} \hat{v}_D(i_D) di_D,$$

while the externally supplied potentials are given by  $\mathcal{P}_E^1 = \mathcal{P}_E^0 = v_E i_L$  and  $\mathcal{P}_B^1 = \mathcal{P}_B^0 = 0$ . Hence, after substituting the latter potentials into (29) yields the differential equations describing the circuit in the form (28)

$$C \frac{dv_C}{dt} = (1 - u)i_L - \frac{1}{R_2} v_C,$$

$$L \frac{di_L}{dt} = v_E - R_1 i_L - (1 - u)[\hat{v}_D(i_L) + v_C].$$

The PHD equations are easily found by noting that the total stored energy equals  $\mathcal{H}(q_C, \phi_L) = (1/2C)q_C^2 + (1/2L)\phi_L^2$ ,

the interconnection (or Dirac) structure satisfies (12), with the modification  $J = J(u)$ , i.e.,

$$J(u) = \begin{bmatrix} 0 & 1-u \\ -1+u & 0 \end{bmatrix}.$$

It should be pointed out the inclusion of the diode in the PHD description is restricted to diodes having a characteristic satisfying Morse Lemma, i.e.,  $\hat{v}_d(0) = 0$ . In that case the diode can be included as

$$\hat{v}_d(\varphi_L) = \frac{\partial \hat{v}_d}{\partial i_L}(i_L) i_L \Big|_{i_L = \partial \mathcal{H} / \partial \varphi_L},$$

using a negative feedback  $\gamma_R = -\hat{v}_d(\varphi_L)$ , like in Section 5.1.

**Remark 12.** For completeness, we state without prove that the switched BM equations are also closely related to the average PWM (pulse-width modulation) models.<sup>3</sup> Such PWM policy may be specified as follows Ortega et al. (1998)

$$u(t) = \begin{cases} 1 & \text{for } t_k \leq t < t_k + \mu(t_k)T, \\ 0 & \text{for } t_k + \mu(t_k)T \leq t < t_k + T, \end{cases}$$

for  $t_{k+1} = t_k + T$ ,  $k=0, 1, 2, \dots$ , where  $t_k$  represents a sampling instant,  $T$  is the fixed sampling period (duty cycle), and  $\mu(\cdot)$  is the duty ratio function of the switch limited to take place in the closed interval  $\mu(\cdot) \in [0, 1]$ . For (29) this means that  $x$  is replaced by the average state  $\bar{x}$ , representing the average capacitor voltages and inductor currents, and the discrete control vector  $u$  is replaced by its duty ratio function vector  $\mu$ . We thus have the following consistency conditions on the potential functions

$$\mathcal{P}^\mu(\bar{x})|_{\mu=1} = \mathcal{P}^1(x),$$

$$\mathcal{P}^\mu(\bar{x})|_{\mu=0} = \mathcal{P}^0(x),$$

where  $\mathcal{P}^1(\bar{x})$  is the potential function for the extreme saturation value  $\mu = 1$ , and  $\mathcal{P}^0(\bar{x})$  is the potential function for the extreme saturation value  $\mu = 0$ . Note that  $\mathcal{P}^\mu(\bar{x})$  can be considered as a weighted ratio, with weighting parameter  $\mu$ , between  $\mathcal{P}^1(\bar{x})$  and  $\mathcal{P}^0(\bar{x})$ .

**Remark 13.** The BM equations are thus shown to be suitable for inclusion of controllable switches. This directly suggests application of similar controller design techniques as developed for PCH and Lagrangian systems with some additional advantages. For example, in contrast to the Lagrangian or Hamiltonian framework, a passivity-based feedback controller (see Jeltsema and Scherpen (2002) and Ortega, Jeltsema, and Scherpen (2002) for some first results regarding control in the BM framework) based on the BM equations requires state measurements in terms of currents and voltages directly. This is a major advantage since they correspond to the commonly used and available sensors

because no complicated and performance degrading (due to unknown parameters) state transformations have to be made.

## 6. Summary and conclusion

In the context of topologically complete electrical circuits we have established a direct connection between the classical Brayton–Moser (BM) equations (*old fish*) and the recently developed port-Hamiltonian (PH) framework with dissipation (PHD) (*new fish*). A full Legendre transform is used to relate the energy and co-energy functions. If the Legendre transformation is bijective, the class of BM and PH equations is the same. However, in e.g. the mechanical domain this will not always be the case, which is easily seen with systems that are subject to gravity. For this class of systems the PH description is ‘easily’ obtained, but there exists no counter part for BM. Furthermore, it has been shown that if the BM equations are expressed in terms of the energy variables (capacitor charges and inductor fluxes) they lead to a time-differentiated version of the PH equations. As a result, many important properties can be exchanged between the two frameworks. We have developed a novel systematic procedure to deal with networks containing inductor-only cutsets and capacitor-only loops. This was inspired by the concept of implicit mechanical systems using Lagrange multipliers and kinematic constraints. Moreover, the mixed-potential function as defined by Brayton and Moser was shown, in partitioned form, to be derivable from an PH external port point of view. For that reason, we may call the BM equations a co-Hamiltonian system with dissipation (PH\*D).

During the developments in this paper we have seen that both frameworks exhibit inherent limitations. As a result, one sometimes has to make a choice between one framework or the other. In case a given circuit contains non-bijective charge-controlled capacitors, one takes the PH equations, while in case of non-bijective current- or voltage controlled resistors, or a highly nonlinear model to be used for feedback control, one should rather vote for the BM equations. Although it is not discussed in this paper, the only framework which does not have such restriction is the generalized Lagrangian description as proposed by Kwatny et al. (1982). However, the drawback of that approach is that the calculations to arrive at the dynamic equations become quite involved, especially when highly nonlinear or large networks have to be analyzed.

In order to use the BM framework to model switched-mode circuits, we have accommodated the BM equations for the inclusion of switching functions. This allows us to apply the well-known passivity techniques for controller design in a BM setting. The main advantage is that the states required for feedback are then directly expressed in terms of voltages and/or current which are immediately available through standard sensors.

<sup>3</sup> See Ortega et al. (1998) for a detailed discussion on this subject in the Lagrangian and Hamiltonian framework.

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